応用システム工学
第五回　回帰分析

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重回帰分析

線形重回帰

- 1つの目的変数yに対する複数(p個)の説明変数$x_1, x_2, \cdots, x_p$
  - 線形重回帰モデル

$$y_i = a_0 + a_1 x_{1i} + a_2 x_{2i} + \cdots + a_p x_{pi} + e_i \qquad (i = 1, 2, \cdots, n)$$

	目的変数:y	説明変数:x	予測誤差
1	y_1	$x_{11}, x_{21}, \cdots, x_{p1}$	$e_1 = y_1 - (a_0 + a_1 x_{11} + a_2 x_{21} + \cdots + a_p x_{p1})$
2	y_2	$x_{12}, x_{22}, \cdots, x_{p2}$	$e_2 = y_2 - (a_0 + a_1 x_{12} + a_2 x_{22} + \cdots + a_p x_{p2})$
…			
i	y_i	$x_{1i}, x_{2i}, \cdots, x_{pi}$	$e_i = y_i - (a_0 + a_1 x_{1i} + a_2 x_{2i} + \cdots + a_p x_{pi})$
…			
n	y_n	$x_{1n}, x_{2n}, \cdots, x_{pn}$	$e_n = y_n - (a_0 + a_1 x_{1n} + a_2 x_{2n} + \cdots + a_p x_{pn})$

線形重回帰

- 単回帰　(x,y)平面
- 重回帰　(x_1,x_2,・・・,x_p,y) (p+1)次元空間
  – P次元超平面

$$y = a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_p x_p$$

  – 説明変数x_{1i},x_{2i},・・・,x_{pi}から目的変数y_iを予測
    - 予測誤差

$$e_i = y_i - \left(a_0 + a_1 x_{1i} + a_2 x_{2i} + \cdots + a_p x_{pi}\right)$$

    - 予測誤差の平方和

$$\sum_{i=1}^{n} e_i^{\ 2} = \sum_{i=1}^{n} \left\{y_i - \left(a_0 + a_1 x_{1i} + a_2 x_{2i} + \cdots + a_p x_{pi}\right)\right\}^2$$

線形重回帰

- 予測誤差の平方和の最小化

$$F\left(a_0, a_1, \cdots, a_p\right) = \sum_{i=1}^{n} e_i^{\ 2}$$

$$= \sum_{i=1}^{n} \left\{y_i - \left(a_0 + a_1 x_{1i} + a_2 x_{2i} + \cdots + a_p x_{pi}\right)\right\}^2$$

$$\begin{cases} \dfrac{\partial}{\partial a_0} F\left(a_0, a_1, \cdots, a_p\right) = 0 \\ \dfrac{\partial}{\partial a_1} F\left(a_0, a_1, \cdots, a_p\right) = 0 \\ \qquad \vdots \\ \dfrac{\partial}{\partial a_p} F\left(a_0, a_1, \cdots, a_p\right) = 0 \end{cases}$$

線形重回帰

$$\frac{\partial}{\partial a_0}F\left(a_0,a_1,\cdots,a_p\right)=\frac{\partial}{\partial a_0}\sum_{i=1}^{n}\left\{y_i-\left(a_0+a_1x_{1i}+a_2x_{2i}+\cdots+a_px_{pi}\right)\right\}^2$$

$$=-2\sum_{i=1}^{n}\left\{y_i-\left(a_0+a_1x_{1i}+a_2x_{2i}+\cdots+a_px_{pi}\right)\right\}=0$$

$$\frac{\partial}{\partial a_1}F\left(a_0,a_1,\cdots,a_p\right)=\frac{\partial}{\partial a_1}\sum_{i=1}^{n}\left\{y_i-\left(a_0+a_1x_{1i}+a_2x_{2i}+\cdots+a_px_{pi}\right)\right\}^2$$

$$=-2\sum_{i=1}^{n}x_{1i}\left\{y_i-\left(a_0+a_1x_{1i}+a_2x_{2i}+\cdots+a_px_{pi}\right)\right\}=0$$

$$\frac{\partial}{\partial a_p}F\left(a_0,a_1,\cdots,a_p\right)=\frac{\partial}{\partial a_p}\sum_{i=1}^{n}\left\{y_i-\left(a_0+a_1x_{1i}+a_2x_{2i}+\cdots+a_px_{pi}\right)\right.$$

$$=-2\sum_{i=1}^{n}x_{pi}\left\{y_i-\left(a_0+a_1x_{1i}+a_2x_{2i}+\cdots+a_px_{pi}\right)\right\}=0$$

線形重回帰

$$\begin{cases}\displaystyle\sum_{i=1}^{n}\left\{y_i-\left(a_0+a_1x_{1i}+a_2x_{2i}+\cdots+a_px_{pi}\right)\right\}=0\\ \displaystyle\sum_{i=1}^{n}x_{1i}\left\{y_i-\left(a_0+a_1x_{1i}+a_2x_{2i}+\cdots+a_px_{pi}\right)\right\}=0\\ \vdots\\ \displaystyle\sum_{i=1}^{n}x_{pi}\left\{y_i-\left(a_0+a_1x_{1i}+a_2x_{2i}+\cdots+a_px_{pi}\right)\right\}=0\end{cases}$$

線形重回帰

- 正規方程式:残差二乗和を最小にする推定値を与える方程式→a_0,a_1,･･･,a_pの連立方程式

$$
\begin{cases}
a_0 n + a_1 \sum_{i=1}^{n} x_{1i} + a_2 \sum_{i=1}^{n} x_{2i} + \cdots + a_p \sum_{i=1}^{n} x_{pi} = \sum_{i=1}^{n} y_i \\
a_0 \sum_{i=1}^{n} x_{1i} + a_1 \sum_{i=1}^{n} {x_{1i}}^2 + a_2 \sum_{i=1}^{n} x_{1i} x_{2i} + \cdots + a_p \sum_{i=1}^{n} x_{1i} x_{pi} = \sum_{i=1}^{n} x_{1i} y_i \\
\vdots \\
a_0 \sum_{i=1}^{n} x_{pi} + a_1 \sum_{i=1}^{n} x_{pi} x_{1i} + a_2 \sum_{i=1}^{n} x_{pi} x_{2i} + \cdots + a_p \sum_{i=1}^{n} x_{pi} x_{pi} = \sum_{i=1}^{n} x_{pi} y_i
\end{cases}
$$

線形重回帰

$$
\begin{bmatrix}
n & \sum_{i=1}^{n} x_{1i} & \sum_{i=1}^{n} x_{2i} & \cdots & \sum_{i=1}^{n} x_{pi} \\
\sum_{i=1}^{n} x_{1i} & \sum_{i=1}^{n} {x_{1i}}^2 & \sum_{i=1}^{n} x_{1i} x_{2i} & & \sum_{i=1}^{n} x_{1i} x_{pi} \\
\vdots & & & & \vdots \\
\sum_{i=1}^{n} x_{pi} & \sum_{i=1}^{n} x_{pi} x_{1i} & \sum_{i=1}^{n} x_{pi} x_{2i} & & \sum_{i=1}^{n} x_{pi} x_{pi}
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_p \end{bmatrix}
=
\begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_{1i} y_i \\ \vdots \\ \sum_{i=1}^{n} x_{pi} y_i \end{bmatrix}
$$

線形重回帰

$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} n & \sum_{i=1}^{n} x_{1i} & \sum_{i=1}^{n} x_{2i} & \cdots & \sum_{i=1}^{n} x_{pi} \\ \sum_{i=1}^{n} x_{1i} & \sum_{i=1}^{n} x_{1i}{}^2 & \sum_{i=1}^{n} x_{1i}x_{2i} & & \sum_{i=1}^{n} x_{1i}x_{pi} \\ \vdots & & & & \vdots \\ \sum_{i=1}^{n} x_{pi} & \sum_{i=1}^{n} x_{pi}x_{1i} & \sum_{i=1}^{n} x_{pi}x_{2i} & & \sum_{i=1}^{n} x_{pi}x_{pi} \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_{1i}y_i \\ \vdots \\ \sum_{i=1}^{n} x_{pi}y_i \end{bmatrix}$$

$$a_0, a_1, \cdots, a_p \Rightarrow \hat{a}_0, \hat{a}_1, \cdots, \hat{a}_p$$

分散共分散で表すことを考える

線形重回帰

- $x_1,\cdots,x_p$の分散共分散行列

$$V = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1l} & \cdots & s_{1p} \\ s_{21} & s_{22} & & & & s_{2p} \\ \vdots & & & & & \vdots \\ s_{j1} & s_{j2} & \cdots & s_{jl} & \cdots & s_{jp} \\ \vdots & & & & & \vdots \\ s_{p1} & s_{p2} & \cdots & s_{pl} & \cdots & s_{pp} \end{bmatrix}$$

$$s_{jl} = \frac{1}{n}\sum_{i=1}^{n}\left(x_{ji} - \bar{x}_j\right)\left(x_{li} - \bar{x}_l\right)$$

$$j, l = 1,2,\cdots,p$$

- yと$x_1,\cdots,x_p$の共分散

$$s_{yj} = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)\left(x_{ji} - \bar{x}_j\right) \quad j = 1,2,\cdots,p$$

線形重回帰
分散共分散行列であらわす

- 正規方程式の一行目

$$\hat{a}_0 n + \hat{a}_1 \sum_{i=1}^{n} x_{1i} + \hat{a}_2 \sum_{i=1}^{n} x_{2i} + \cdots + \hat{a}_p \sum_{i=1}^{n} x_{pi} = \sum_{i=1}^{n} y_i$$

$$\hat{a}_0 = \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{1}{n}\left(\hat{a}_1 \sum_{i=1}^{n} x_{1i} + \hat{a}_2 \sum_{i=1}^{n} x_{2i} + \cdots + \hat{a}_p \sum_{i=1}^{n} x_{pi} \right)$$
$$= \bar{y} - \left(\hat{a}_1 \bar{x}_1 + \hat{a}_2 \bar{x}_2 + \cdots + \hat{a}_p \bar{x}_p \right)$$

$$\hat{a}_0 = \bar{y} - \left(\hat{a}_1 \bar{x}_1 + \hat{a}_2 \bar{x}_2 + \cdots + \hat{a}_p \bar{x}_p \right)$$

  - 正規方程式のa_0に代入

線形重回帰

$$\begin{cases} \sum_{i=1}^{n} \left\{ y_i - \left(\left[\bar{y} - \left(\hat{a}_1 \bar{x}_1 + \hat{a}_2 \bar{x}_2 + \cdots + \hat{a}_p \bar{x}_p \right) \right] + \hat{a}_1 x_{1i} + \hat{a}_2 x_{2i} + \cdots + \hat{a}_p x_{pi} \right) \right\} = 0 \\ \sum_{i=1}^{n} x_{1i} \left\{ y_i - \left(\left[\bar{y} - \left(\hat{a}_1 \bar{x}_1 + \hat{a}_2 \bar{x}_2 + \cdots + \hat{a}_p \bar{x}_p \right) \right] + \hat{a}_1 x_{1i} + \hat{a}_2 x_{2i} + \cdots + \hat{a}_p x_{pi} \right) \right\} = 0 \\ \vdots \\ \sum_{i=1}^{n} x_{ji} \left\{ y_i - \left(\left[\bar{y} - \left(\hat{a}_1 \bar{x}_1 + \hat{a}_2 \bar{x}_2 + \cdots + \hat{a}_p \bar{x}_p \right) \right] + \hat{a}_1 x_{1i} + \hat{a}_2 x_{2i} + \cdots + \hat{a}_p x_{pi} \right) \right\} = 0 \\ \vdots \\ \sum_{i=1}^{n} x_{pi} \left\{ y_i - \left(\left[\bar{y} - \left(\hat{a}_1 \bar{x}_1 + \hat{a}_2 \bar{x}_2 + \cdots + \hat{a}_p \bar{x}_p \right) \right] + \hat{a}_1 x_{1i} + \hat{a}_2 x_{2i} + \cdots + \hat{a}_p x_{pi} \right) \right\} = 0 \end{cases}$$

線形重回帰

$$
\begin{cases}
\sum_{i=1}^{n}\left\{(y_i-\bar{y})-\hat{a}_1(x_{1i}-\bar{x}_1)-\hat{a}_2(x_{2i}-\bar{x}_2)-\cdots-\hat{a}_p(x_{pi}-\bar{x}_p)\right\}=0 \\
\sum_{i=1}^{n}x_{1i}\left\{(y_i-\bar{y})-\hat{a}_1(x_{1i}-\bar{x}_1)-\hat{a}_2(x_{2i}-\bar{x}_2)-\cdots-\hat{a}_p(x_{pi}-\bar{x}_p)\right\}=0 \\
\vdots \\
\sum_{i=1}^{n}x_{ji}\left\{(y_i-\bar{y})-\hat{a}_1(x_{1i}-\bar{x}_1)-\hat{a}_2(x_{2i}-\bar{x}_2)-\cdots-\hat{a}_p(x_{pi}-\bar{x}_p)\right\}=0 \\
\vdots \\
\sum_{i=1}^{n}x_{pi}\left\{(y_i-\bar{y})-\hat{a}_1(x_{1i}-\bar{x}_1)-\hat{a}_2(x_{2i}-\bar{x}_2)-\cdots-\hat{a}_p(x_{pi}-\bar{x}_p)\right\}=0
\end{cases}
$$

各行j=1~pから，j=0行×$\bar{x}_j$を引く　　　共通

線形重回帰

$$
\begin{cases}
\sum_{i=1}^{n}(x_{1i}-\bar{x}_1)\left\{(y_i-\bar{y})-\hat{a}_1(x_{1i}-\bar{x}_1)-\hat{a}_2(x_{2i}-\bar{x}_2)-\cdots-\hat{a}_p(x_{pi}-\bar{x}_p)\right\}=0 \\
\vdots \\
\sum_{i=1}^{n}(x_{ji}-\bar{x}_j)\left\{(y_i-\bar{y})-\hat{a}_1(x_{1i}-\bar{x}_1)-\hat{a}_2(x_{2i}-\bar{x}_2)-\cdots-\hat{a}_p(x_{pi}-\bar{x}_p)\right\}=0 \\
\vdots \\
\sum_{i=1}^{n}(x_{pi}-\bar{x}_p)\left\{(y_i-\bar{y})-\hat{a}_1(x_{1i}-\bar{x}_1)-\hat{a}_2(x_{2i}-\bar{x}_2)-\cdots-\hat{a}_p(x_{pi}-\bar{x}_p)\right\}=0
\end{cases}
$$

各行の両辺をnで割ると

線形重回帰

$$\begin{cases} s_{y1} - \hat{a}_1 s_{11} - \hat{a}_2 s_{12} - \cdots - \hat{a}_p s_{1p} = 0 \\ \vdots \\ s_{yj} - \hat{a}_1 s_{j1} - \hat{a}_2 s_{j2} - \cdots - \hat{a}_p s_{jp} = 0 \\ \vdots \\ s_{yp} - \hat{a}_1 s_{p1} - \hat{a}_2 s_{p2} - \cdots - \hat{a}_p s_{pp} = 0 \end{cases}$$

$$\begin{cases} \hat{a}_1 s_{11} + \hat{a}_2 s_{12} + \cdots + \hat{a}_p s_{1p} = s_{y1} \\ \vdots \\ \hat{a}_1 s_{j1} + \hat{a}_2 s_{j2} + \cdots + \hat{a}_p s_{jp} = s_{yj} \\ \vdots \\ \hat{a}_1 s_{p1} + \hat{a}_2 s_{p2} + \cdots + \hat{a}_p s_{pp} = s_{yp} \end{cases}$$

線形重回帰

$$\begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1p} \\ \vdots & & & \\ s_{j1} & s_{j2} & & s_{jp} \\ \vdots & & & \\ s_{p1} & s_{p2} & \cdots & s_{pp} \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \vdots \\ \hat{a}_p \end{bmatrix} = \begin{bmatrix} s_{y1} \\ \vdots \\ s_{yj} \\ \vdots \\ s_{yp} \end{bmatrix}$$

分散共分散行列V　　　yのxに対する共分散

別途 $\hat{a}_0 = \bar{y} - \left(\hat{a}_1 \bar{x}_1 + \hat{a}_2 \bar{x}_2 + \cdots + \hat{a}_p \bar{x}_p\right)$ で $\hat{a}_0$を求めればよい

目的変数yの，説明変数$x_1, x_2, \cdots, x_p$に対する線形重回帰式

$$y = \hat{a}_0 + \hat{a}_1 x_1 + \hat{a}_2 x_2 + \cdots + \hat{a}_p x_p$$

重回帰係数 $\hat{a}_j \quad j = 1,2,\cdots,p$

重回帰式の予測誤差の標準偏差

- 予測誤差の標準偏差

$$s_e = \sqrt{\frac{1}{n-(p+1)}\sum_{i=1}^{n}(e_i - \bar{e})^2} = \sqrt{\frac{1}{n-(p+1)}\sum_{i=1}^{n}{e_i}^2}$$

  – ただし $\bar{e} = \frac{1}{n}\sum_{i=1}^{n} e_i = \frac{1}{n}\sum_{i=1}^{n}\left\{y_i - \left(\hat{a}_0 + \hat{a}_1 x_{1i} + \hat{a}_2 x_{2i} + \cdots + \hat{a}_p x_{pi}\right)\right\}$

$\hat{a}_0 = \bar{y} - \left(\hat{a}_1 \bar{x}_1 + \hat{a}_2 \bar{x}_2 + \cdots + \hat{a}_p \bar{x}_p\right)$ より

$$\bar{e} = \frac{1}{n}\sum_{i=1}^{n}\left\{y_i - \left[\bar{y} - \left(\hat{a}_1 \bar{x}_1 + \hat{a}_2 \bar{x}_2 + \cdots + \hat{a}_p \bar{x}_p\right) + \hat{a}_1 x_{1i} + \hat{a}_2 x_{2i} + \cdots + \hat{a}_p x_{pi}\right]\right\}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left\{\left(y_i - \bar{y}\right) - \hat{a}_1\left(x_{1i} - \bar{x}_1\right) \cdots \hat{a}_p\left(x_{pi} - \bar{x}_p\right)\right\} = 0$$

$\frac{\partial}{\partial a_0} F\left(a_0, a_1, \cdots, a_p\right) = 0$ に一致

重相関係数

- 重回帰式による予測値

$$Y_i = \hat{a}_0 + \hat{a}_1 \bar{x}_1 + \hat{a}_2 \bar{x}_2 + \cdots + \hat{a}_p \bar{x}_p$$

- 予測値Yiは説明変数で表される
- 目的変数yと予測値Yの単相関係数 r_{yY}
  →目的変数yと説明変数x_1,x_2,・・・,x_pの重相関係数

$$r_{y\cdot 12\cdots p} = \frac{s_{yY}}{\sqrt{s_{yy} s_{YY}}} = \frac{\frac{1}{n}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)\left(Y_i - \bar{Y}\right)}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2 \frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2}}$$

  – 回帰平面(p+1)次元に近いかどうかを表す

重相関係数

- 重相関係数のとる範囲は？

$$r_{y\cdot 12\cdots p}=\frac{s_{yY}}{\sqrt{s_{yy}s_{YY}}}=\frac{\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})(Y_i-\bar{Y})}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})^2\frac{1}{n}\sum_{i=1}^{n}(Y_i-\bar{Y})^2}}$$

– 分子について考える

$$s_{yY}=\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})(Y_i-\bar{Y})$$

重相関係数

– 目的変数と予測値の平均

$$\bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i \qquad \bar{Y}=\frac{1}{n}\sum_{i=1}^{n}Y_i=\frac{1}{n}\sum_{i=1}^{n}(y_i-e_i)=\frac{1}{n}\sum_{i=1}^{n}y_i=\bar{y}$$

– 分子

$$s_{yY}=\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})(Y_i-\bar{Y})=\frac{1}{n}\sum_{i=1}^{n}(Y_i+e_i-\bar{Y})(Y_i-\bar{Y})$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left\{(Y_i-\bar{Y})^2+e_i(Y_i-\bar{Y})\right\}$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left\{(Y_i-\bar{Y})^2+e_i(a_0+a_1x_{1i}+\cdots+a_px_{pi}-\bar{Y})\right\}$$

$$=\frac{1}{n}\sum_{i=1}^{n}(Y_i-\bar{Y})^2=s_{YY}\geq 0 \quad \Rightarrow \quad r_{y\cdot 12\cdots p}=\frac{s_{yY}}{\sqrt{s_{yy}s_{YY}}}\geq 0$$

重相関係数

– 回帰式の残差平方和を最小にする係数の条件

$$\frac{\partial F}{\partial a_0}=\frac{\partial}{\partial a_0}\sum_{i=1}^{n}\left\{y_i-\left(a_0+a_1x_{1i}+a_2x_{2i}+\cdots+a_px_{pi}\right)\right\}^2$$

$$=-2\sum_{i=1}^{n}\left\{y_i-\left(a_0+a_1x_{1i}+a_2x_{2i}+\cdots+a_px_{pi}\right)\right\}=\sum_{i=1}^{n}e_i=0$$

$$\frac{\partial F}{\partial a_j}=\frac{\partial}{\partial a_j}\sum_{i=1}^{n}\left\{y_i-\left(a_0+a_1x_{1i}+a_2x_{2i}+\cdots+a_px_{pi}\right)\right\}^2$$

$$=-2\sum_{i=1}^{n}x_{ji}\left\{y_i-\left(a_0+a_1x_{1i}+\cdots+a_px_{pi}\right)\right\}=-2\sum_{i=1}^{n}x_{ji}e_i=0$$

重相関係数

- 両辺二乗 $r_{y\cdot 12\cdots p}{}^2=\frac{s_{yY}{}^2}{s_{yy}s_{YY}}=\frac{s_{YY}{}^2}{s_{yy}s_{YY}}=\frac{s_{YY}}{s_{yy}}$
- 目的変数・予測値の分散の関係を求める

$$s_{yy}=\frac{1}{n}\sum_{i=1}^{n}\left(y_i-\bar{y}\right)^2=\frac{1}{n}\sum_{i=1}^{n}\left(Y_i+e_i-\bar{Y}\right)^2$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left\{\left(Y_i-\bar{Y}\right)^2+2e_i\left(Y_i-\bar{Y}\right)+e_i{}^2\right\}$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left\{\left(Y_i-\bar{Y}\right)^2+2e_i\left(a_0+a_1x_{1i}+\cdots+a_px_{pi}-\bar{Y}\right)+e_i{}^2\right\}$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left\{\left(Y_i-\bar{Y}\right)^2+e_i{}^2\right\}=s_{YY}+\frac{1}{n}\sum_{i=1}^{n}e_i{}^2$$

重相関係数

$$\frac{1}{n}\sum_{i=1}^{n} e_i^{\ 2} = s_{yy} - s_{YY} = s_{yy}\left(1 - \frac{s_{YY}}{s_{yy}}\right)$$

$$= s_{yy}\left(1 - r_{y\cdot 12\cdots p}^{\ \ 2}\right) \geq 0$$

$$1 - r_{y\cdot 12\cdots p}^{\ \ 2} \geq 0$$

$$-1 \leq r_{y\cdot 12\cdots p} \leq 1$$

$$r_{y\cdot 12\cdots p} = \frac{s_{yY}}{\sqrt{s_{yy}s_{YY}}} \geq 0$$

より

$$0 \leq r_{y\cdot 12\cdots p} \leq 1$$